

The character table of the Hecke algebra  $H_n(q)$  in terms of traces of  
products of Murphy operators

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## Abstract

The traces of the Murphy operators of the Hecke algebra  $H_n(q)$ , and of products of sets of Murphy operators with non-consecutive indices, can be evaluated by a straightforward recursive procedure. These traces are shown to determine all the reduced traces in this algebra, which, in turn, determine all other traces. To illustrate the procedure we obtain the set of reduced traces for  $H_7(q)$  - the lowest order Hecke algebra whose character table has not hitherto been reported. This is preceded by the presentation of an explicit algorithm for the reduction of the trace of an arbitrary element of the Hecke algebra into a linear combination of traces of elements consisting of appropriately defined disjoint cycles; and of a proof, presented in order to make the present article reasonably self-contained, that a reduced trace depends only on the set of lengths of the disjoint cycles that it consists of.

# 1 Introduction

The present article is a sequel to [1], in which the remarkable properties of the fundamental invariant of the Hecke algebra  $H_n(q)$  were elucidated. The sequence of sub-algebras  $H_2(q) \subset H_3(q) \subset \cdots \subset H_n(q)$  gives rise to a sequence of mutually commuting fundamental invariants  $C_2, C_3, \dots, C_n$ . In the generic case, *i.e.*, when  $q$  is neither a root of unity of order up to  $n$ , nor zero, the eigenvalues of  $C_n$ , the fundamental invariant of  $H_n(q)$ , fully characterize the corresponding irreducible representations (irreps). In fact, these eigenvalues are polynomials in  $q$  and  $\frac{1}{q}$ , and the coefficients of the various powers are directly related to the structure of the Young diagram specifying the irrep. Furthermore, the common set of eigenvectors of  $C_2, C_3, \dots, C_n$ , for any *particular* eigenvalue of  $C_n$  but all the consistent eigenvalues of  $C_2, C_3, \dots, C_{n-1}$ , spans the corresponding irreducible invariant subspace. In this context it turns out that the set of (mutually commuting) Murphy operators  $L_2 = C_2, L_3 = C_3 - C_2, \dots, L_n = C_n - C_{n-1}$  have particularly attractive properties, the primary one being that the corresponding set of eigenvalues characterizes in a very convenient and explicit form a specific eigenvector within the basis specified above.

The purpose of the present article is to provide a systematic procedure that enables the evaluation of arbitrary traces in terms of the easily evaluated traces of products of sets of Murphy operators. We begin by presenting an algorithm that effects the reduction of the trace of an arbitrary element of the Hecke algebra into a linear combination of reduced traces. The latter consist of disjoint sequences of consecutive generators. The fact that a reduced trace only depends on the lengths of the disjoint sequences of generators of which it consists was noted by King and Wybourne [2] on the basis of explicit computations for the Hecke algebras  $H_2(q)$  to  $H_6(q)$ , but had in fact been shown in an unpublished doctoral dissertation by Starkey [3]. To make the present article reasonably self-contained we present in section 4 a proof of this property, formulated in three lemmas. The direct evaluation of the traces of the reduced elements of the algebra, without prior explicit construction of the irreps, was the main purpose of ref. [2], which is the most comprehensive treatment of the evaluation of traces of elements of the Hecke algebra that we are aware of, and to which we refer for

a discussion of earlier relevant references. Further closely relevant references are Ram [4] and Ueno and Shibukawa [5]. However, we show in the present article that the systematic investigation of the Murphy operators provides a remarkably simple and convenient means for the evaluation of the reduced traces.

## 2 The Hecke algebra $H_n(q)$

The Hecke algebra  $H_n(q)$  is defined in terms of the generators  $g_1, g_2, \dots, g_{n-1}$  and the relations

$$\begin{aligned} g_i^2 &= (q-1)g_i + q & i &= 1, 2, \dots, n-1 \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} & i &= 1, 2, \dots, n-2 \\ g_i g_j &= g_j g_i & \text{if } |i-j| &\geq 2 \end{aligned} \tag{1}$$

For  $q = 1$  these relations reduce to the generating relations of the symmetric group,  $S_n$ . In particular,  $g_i$  reduces to the transposition  $(i, i+1)$ .

When  $q$  is neither zero nor a  $k$ th root of unity,  $k = 2, 3, \dots, n$ , the irreps of  $H_n(q)$  are labelled by Young diagrams with  $n$  boxes [6, 7].

The Murphy operators are [8, 9]

$$\begin{aligned} L_p &= g_{p-1} + \frac{1}{q} g_{p-2} g_{p-1} g_{p-2} + \frac{1}{q^2} g_{p-3} g_{p-2} g_{p-1} g_{p-2} g_{p-3} \\ &\quad + \dots + \frac{1}{q^{p-2}} g_1 g_2 \dots g_{p-2} g_{p-1} g_{p-2} \dots g_2 g_1 \\ &= \sum_{i=1}^{p-1} \frac{1}{q^{p-1-i}} g_i g_{i+1} \dots g_{p-2} g_{p-1} g_{p-2} \dots g_{i+1} g_i ; \quad p = 2, 3, \dots, n \end{aligned} \tag{2}$$

Any two Murphy operators commute with one another. A state labelled by the sequence of Young diagrams  $\Gamma_2 \subset \Gamma_3 \subset \dots \subset \Gamma_n$  is an eigenstate of all the Murphy operators  $L_2, L_3, \dots, L_n$ , the eigenvalue of  $L_i$  being [8, 9]

$$\{\Gamma_i \setminus \Gamma_{i-1}\}_q \equiv q[\kappa_i - \rho_i]_q ,$$

where  $\kappa_i$  and  $\rho_i$  are the column and row indices, respectively, of the box that has been added to  $\Gamma_{i-1}$  to obtain  $\Gamma_i$ , and  $[x]_q \equiv \frac{q^x - 1}{q - 1}$ .

The fundamental invariant of the Hecke algebra is the sum of the Murphy operators,

$$C_n = \sum_{i=2}^n L_i .$$

Its eigenvalue, corresponding to an irrep  $\Gamma_n$ , is given by

$$\Lambda_{C_n}^{\Gamma_n} = q \sum_{(i,j) \in \Gamma_n} [j-i]_q ,$$

where  $i$  and  $j$  are the row and column indices, respectively, of the boxes comprising the Young diagram that corresponds to  $\Gamma_n$ .

It will be convenient to introduce the shorthand notation

$$(g_i)^{(\ell)} \equiv g_i g_{i+1} \cdots g_{i+\ell-1} \cdots g_{i+1} g_i$$

and

$$(g_i)_\ell \equiv g_i g_{i+1} \cdots g_{i+\ell-1} .$$

Using this notation

$$L_p = \sum_{i=1}^{p-1} \frac{1}{q^{p-1-i}} (g_i)^{(p-i)} = \sum_{j=1}^{p-1} \frac{1}{q^{j-1}} (g_{p-j})^{(j)} .$$

The generalized braiding relation

$$g_i g_{i+1} \cdots g_{j-1} g_j g_{j-1} \cdots g_{i+1} g_i = g_j g_{j-1} \cdots g_{i+1} g_i g_{i+1} \cdots g_{j-1} g_j$$

is easily proved and sometimes useful.

Some remarkable and handy identities that are easily derived are

$$g_i (g_j)_{m-j+1} = (g_j)_{m-j+1} g_{i-1} \quad \text{for } i \leq j-2 ; j+1 \leq i \leq m ; i \geq m+2$$

$$g_i (g_j)^{(m-j+1)} = (g_j)^{(m-j+1)} g_i \quad \text{for } i \leq j-2 ; j+1 \leq i \leq m-1 ; i \geq m+2$$

$$L_{p+1} = \frac{1}{(q-1)q^{p-1}} \left( (g_p g_{p-1} \cdots g_1) (g_1 \cdots g_{p-1} g_p) - q^p \right) \quad (3)$$

From the last identity it follows that

$$L_{p+1} = \frac{1}{q} g_p L_p g_p + g_p .$$

### 3 Reduction of the trace of an arbitrary element of the Hecke algebra

We shall refer to a product of  $\ell$  consecutive generators of the Hecke algebra,  $(g_i)_\ell$ , as a connected sequence of length  $\ell$ , or sometimes, generalizing the corresponding notion from the context of the symmetric group, as a cycle of length  $\ell + 1$ . Two connected sequences are disjoint if the sequence with a lower bottom element has a top element whose index is less than and not adjacent to that of the bottom element of the other sequence. Obviously, two disjoint sequences commute with one another. An element of the Hecke algebra consisting of disjoint cycles will be referred to as a reduced element.

King and Wybourne [2] have demonstrated the reduction of the traces of arbitrary elements in the Hecke algebras  $H_2(q)$  to  $H_6(q)$  into linear combinations of reduced traces, *i.e.*, traces of reduced elements. In the present section we provide an algorithm for the reduction of the trace of an arbitrary element of  $H_n(q)$ .

The following observation will be needed. A trace in which each generator appears at most once can be transformed into the trace of the same set of generators, ordered by increasing indices. In the latter form the disjoint connected sequences are readily identified. At the risk of stating the obvious we note that the transformation we refer to can be carried out recursively, the  $\ell$ th step being

$$tr((\phi)_{\ell-1} D g_\ell E) = tr(D(\phi)_{\ell-1} g_\ell E) = tr((\phi)_{\ell-1} g_\ell E D)$$

where  $D$  and  $E$  contain no generator with index less than  $\ell + 1$ , and all higher generators at most once, and where  $(\phi)_{\ell-1}$  is an ordered product of the generators  $g_1, g_2, \dots, g_{\ell-1}$ , containing each one at most once.

An algorithm that achieves the reduction of the trace of an arbitrary product of generators into a linear combination of reduced traces, in a finite number of steps, is formulated as follows. Assume that we have already transformed the trace of the original product of

generators into a linear combination of traces of the form

$$tr\left((\phi)_{\ell-1}g_{\ell}Ag_{\ell}B\right) \quad (4)$$

where  $A$  is an arbitrary product of arbitrary numbers of generators with indices not less than  $\ell + 1$ , and  $B$  is an arbitrary product of arbitrary numbers of generators with indices not less than  $\ell$ .

- If  $g_{\ell+1}$  does not appear within  $A$ , the two  $g_{\ell}$  factors can be brought together, and reduced.
- If  $g_{\ell+1}$  appears within  $A$  precisely once, the two  $g_{\ell}$  factors can be carried to its sides and the braiding relation used to reduce the number of  $g_{\ell}$  factors by one.
- If  $g_{\ell+1}$  appears within  $A$  more than once, the leftmost segment of the form  $g_{\ell+1}A'g_{\ell+1}$  within  $A$  should be treated following the analogue of the appropriate one of the steps that are presently being described. Once the number of  $g_{\ell+1}$  factors between the first two  $g_{\ell}$  factors has been reduced to zero or one, the appropriate mode of reducing the number of  $g_{\ell}$  factors should be applied.

Each one of the traces thus obtained can be brought back to the form of eq. 4 with redefined  $A$  and  $B$ , possibly by appropriate cyclic transformations within each trace, or transposition of commuting factors.

The whole procedure should continue until all the traces generated contain  $g_{\ell}$  at most once, at which stage each trace should be brought to the form

$$tr\left((\phi)_{\ell}g_{\ell+1}Ag_{\ell+1}B\right) ,$$

with appropriately redefined  $A$  and  $B$ , now containing factors with indices at least  $\ell + 2$  and  $\ell + 1$ , respectively.

Note that the starting point of the iteration is a trace of the form  $tr\left(g_1Ag_1B\right)$  where we have interpreted  $(\phi)_0$  to be an empty sequence, that is equal to the identity, and where  $A$  does not contain  $g_1$ .

Since the number of generators is finite the algorithm terminates in a finite number of iterations. The resulting expression is a linear combination of reduced traces.

## 4 Traces of reduced elements with common cycle structure

We shall proceed by proving the following three *Lemmas*.

LEMMA 1: *The trace of a connected sequence of a given length, in any particular irrep, depends only on the length of the sequence.*

PROOF: First, note that for obvious reasons the first index in a connected sequence of length  $\ell$  must satisfy  $1 \leq i \leq n - \ell$ .

Now,

$$\begin{aligned}
tr(g_{i+\ell}(g_i)_\ell g_{i+\ell}) &= (q-1)tr((g_i)_{\ell+1}) + q tr((g_i)_\ell) \\
&= tr(g_i g_{i+1} \cdots g_{i+\ell} g_{i+\ell-1} g_{i+\ell}) = tr(g_i g_{i+1} \cdots g_{i+\ell-1} g_{i+\ell} g_{i+\ell-1}) = \\
&= tr(g_{i+\ell-1} g_i g_{i+1} \cdots g_{i+\ell-1} g_{i+\ell}) = tr(g_i g_{i+1} \cdots g_{i+\ell-1} g_{i+\ell-2} g_{i+\ell-1} g_{i+\ell}) = \\
&= tr(g_i g_{i+1} \cdots g_{i+\ell-2} g_{i+\ell-1} g_{i+\ell-2} g_{i+\ell}) = tr(g_i g_{i+1} \cdots g_{i+\ell-2} g_{i+\ell-3} g_{i+\ell-2} g_{i+\ell-1} g_{i+\ell}) \\
&= \cdots = \\
&= tr(g_{i+1} g_i g_{i+1} g_{i+2} \cdots g_{i+\ell-1} g_{i+\ell}) = tr(g_i g_{i+1} g_i g_{i+2} \cdots g_{i+\ell-1} g_{i+\ell}) = \\
&= (q-1)tr((g_i)_{\ell+1}) + q tr((g_{i+1})_\ell) \quad ; \quad i = 1, 2, \dots, n - \ell - 1.
\end{aligned}$$

Hence,

$$tr((g_i)_\ell) = tr((g_{i+1})_\ell) \quad ; \quad i = 1, 2, \dots, n - \ell - 1,$$

which is a statement of the *Lemma*.

LEMMA 2: *The trace of a product of two disjoint connected sequences, in any particular irrep, depends only on the lengths of the two sequences.*



PROOF: Consider the trace

$$\text{tr}\left((g_i)_\ell (g_j)_m\right)_\Gamma,$$

where  $i + \ell < j$ . First, note that using the argument of *Lemma 1* the indices of each connected sequence in this trace can be shifted as follows:  $i$  can be shifted to any value between 1 and  $j - \ell - 1$  without changing  $j$ . Independently,  $j$  can be shifted to any value between  $i + \ell + 1$  and  $n - m - 1$  without changing  $i$ .

Now, write the above trace in the equivalent form

$$\text{tr}\left((g_1)_\ell (g_{n-m})_m\right)_\Gamma,$$

*i.e.*, shift  $i$  to the bottom and  $j$  to the top. We now show that the trace remains unchanged if the two connected sequences are transposed, so that the sequence of length  $\ell$  would consist of generators with higher indices than the sequence of length  $m$ . Note the automorphism  $g_i \leftrightarrow \tilde{g}_i \equiv g_{n-i}$ ,  $i = 1, 2, \dots, n - 1$ , and consider some irrep  $\Gamma : g_i \mapsto D_i$ . Obviously,  $\tilde{\Gamma} : g_i \mapsto \tilde{D}_i \equiv D_{n-i}$  is an equivalent irrep. Let  $U$  be the transformation matrix from the first to the second irrep, *i.e.*,  $D_i = U \tilde{D}_i U^{-1} = U D_{n-i} U^{-1}$ . It follows that

$$\begin{aligned} \text{tr}\left((g_1)_\ell (g_{n-m})_m\right)_\Gamma &= \text{tr}(D_1 D_2 \cdots D_\ell D_{n-m} D_{n-m+1} \cdots D_{n-1}) = \\ &= \text{tr}(U D_{n-1} U^{-1} U D_{n-2} U^{-1} \cdots U D_{n-\ell} U^{-1} U D_m U^{-1} U D_{m-1} U^{-1} \cdots U D_1 U^{-1}) = \\ &= \text{tr}(U D_{n-1} D_{n-2} \cdots D_{n-\ell} D_m D_{m-1} \cdots D_1 U^{-1}) = \\ &= \text{tr}(g_{n-1} g_{n-2} \cdots g_{n-\ell} g_m g_{m-1} \cdots g_1)_\Gamma = \text{tr}\left((g_1)_m (g_{n-\ell})_\ell\right)_\Gamma \end{aligned}$$

which proves the *Lemma*.

**LEMMA 3:** *The trace of a product of any number of disjoint connected sequences, in any particular irrep, depends only on the set of lengths of the different connected sequences.*

PROOF: We shall proceed by an induction on the number  $\kappa$  of disjoint connected sequences. From *Lemma 2* it follows that the present *Lemma* is true for  $\kappa = 2$ . Assume that it is true for  $\kappa - 1$ .

We shall denote by  $\langle \cdots \rangle_{\Gamma_i \subset \Gamma_{i+1} \subset \cdots \subset \Gamma_j}$  the diagonal matrix element of  $\cdots$  corresponding to any one of the basis vectors specified by  $\Gamma_i \subset \Gamma_{i+1} \subset \cdots \subset \Gamma_j$ . The suppression of

the subsequence  $\Gamma_2 \subset \Gamma_3 \subset \cdots \subset \Gamma_{i-1}$  in the symbol specifying the basis vector implies that the matrix element in question does not depend on that subsequence. Summation over  $(\Gamma_i \subset \Gamma_{i+1} \subset \cdots \subset \Gamma_{j-1}) \subset \Gamma_j$  means that  $\Gamma_i \subset \Gamma_{i+1} \subset \cdots \subset \Gamma_{j-1}$  obtain all the combinations of values consistent with the given  $\Gamma_j$ .

Given

$$\tau \equiv \text{tr} \left( (g_{i_1})_{\ell_1} (g_{i_2})_{\ell_2} \cdots (g_{i_{\kappa-1}})_{\ell_{\kappa-1}} (g_{i_\kappa})_{\ell_\kappa} \right)_{\Gamma_n}$$

where  $1 \leq i_1, i_1 + \ell_1 < i_2, i_2 + \ell_2 < i_3, \cdots, i_{\kappa-1} + \ell_{\kappa-1} < i_\kappa, i_\kappa + \ell_\kappa \leq n$ , we begin by shifting the last connected sequence to the top, obtaining

$$\begin{aligned} \tau &= \text{tr} \left( (g_{i_1})_{\ell_1} (g_{i_2})_{\ell_2} \cdots (g_{i_{\kappa-1}})_{\ell_{\kappa-1}} (g_{n-\ell_\kappa-1})_{\ell_\kappa} \right)_{\Gamma_n} = \\ &= \sum_{(\Gamma_2 \subset \Gamma_3 \subset \cdots \subset \Gamma_{n-1}) \subset \Gamma_n} \left\langle (g_{i_1})_{\ell_1} (g_{i_2})_{\ell_2} \cdots (g_{i_{\kappa-1}})_{\ell_{\kappa-1}} (g_{n-\ell_\kappa-1})_{\ell_\kappa} \right\rangle_{\Gamma_2 \subset \Gamma_3 \subset \cdots \subset \Gamma_{n-1} \subset \Gamma_n} \end{aligned}$$

The operation of  $g_i$  on the state labelled by

$$\Gamma_2 \subset \Gamma_3 \subset \cdots \subset \Gamma_{i-1} \subset \Gamma_i \subset \Gamma_{i+1} \subset \cdots \subset \Gamma_n$$

can only affect the Young diagram  $\Gamma_{i-1}$ , [10], so that

$\langle (g_i)_\ell \rangle_{\Gamma_2 \subset \Gamma_3 \subset \cdots \subset \Gamma_n} = \langle (g_i)_\ell \rangle_{\Gamma_{i-1} \subset \Gamma_i \subset \cdots \subset \Gamma_{i+\ell-2}}$ . Therefore,

$$\begin{aligned} \tau &= \sum_{(\Gamma_{n-\ell_\kappa-2} \subset \Gamma_{n-\ell_\kappa-1} \subset \cdots \subset \Gamma_{n-1}) \subset \Gamma_n} \left\{ \sum_{(\Gamma_2 \subset \Gamma_3 \subset \cdots \subset \Gamma_{n-\ell_\kappa-3}) \subset \Gamma_{n-\ell_\kappa-2}} \left\langle (g_{i_1})_{\ell_1} (g_{i_2})_{\ell_2} \cdots (g_{i_{\kappa-1}})_{\ell_{\kappa-1}} \right\rangle_{\Gamma_2 \subset \Gamma_3 \subset \cdots \subset \Gamma_{n-\ell_\kappa-3} \subset \Gamma_{n-\ell_\kappa-2}} \right\} \\ &\quad \times \left\langle (g_{n-\ell_\kappa-1})_{\ell_\kappa} \right\rangle_{\Gamma_{n-\ell_\kappa-2} \subset \Gamma_{n-\ell_\kappa-1} \subset \cdots \subset \Gamma_{n-1} \subset \Gamma_n} \\ &= \sum_{(\Gamma_{n-\ell_\kappa-2} \subset \Gamma_{n-\ell_\kappa-1} \subset \cdots \subset \Gamma_{n-1}) \subset \Gamma_n} \text{tr} \left( (g_{i_1})_{\ell_1} (g_{i_2})_{\ell_2} \cdots (g_{i_{\kappa-1}})_{\ell_{\kappa-1}} \right)_{\Gamma_{n-\ell_\kappa-2}} \left\langle (g_{n-\ell_\kappa-1})_{\ell_\kappa} \right\rangle_{\Gamma_{n-\ell_\kappa-2} \subset \Gamma_{n-\ell_\kappa-1} \subset \cdots \subset \Gamma_{n-1} \subset \Gamma_n} \end{aligned}$$

Now, within the trace of the product of the first  $\kappa - 1$  disjoint connected sequences the induction hypothesis allows arbitrary permutations of disjoint sequences as well as arbitrary shifts of indices that respect the disjointness conditions. Performing such a combination of

permutations and shifts we establish the *Lemma*, except for permutations involving the top connected sequence. Permutations of that kind can be performed using the argument of the proof of *Lemma 2* to transpose the top and bottom connected sequences in the original form of  $\tau$ . This proves the *Lemma*.

The simplest non-trivial applications of *Lemmas 1, 2, and 3* are the identities  $tr(g_1) = tr(g_2)$ ,  $tr(g_1 g_2 g_4) = tr(g_1 g_3 g_4)$ , and  $tr(g_1 g_3 g_4 g_6) = tr(g_1 g_3 g_5 g_6)$ , respectively.

## 5 Some basic reduction formulas

In the following section we demonstrate that the derivation of expressions for arbitrary reduced traces in terms of traces of products of Murphy operators requires the reduction of traces of the form

$$tr\left((g_1)_{\ell_1}(g_{m_2})_{\ell_2} \cdots (g_{m_\kappa})_{\ell_\kappa}(g_i)^{(p)}\right),$$

where  $m_j = \ell_1 + \ell_2 + \cdots + \ell_{j-1} + j$  and where  $(g_1)_{\ell_1}(g_{m_2})_{\ell_2} \cdots (g_{m_\kappa})_{\ell_\kappa}$  are  $\kappa$  disjoint cycles. The index  $i$  can assume any value, but  $i + p - 1 > m_\kappa + \ell_\kappa - 1$ . In fact, the complete range of cases is covered by  $1 \leq i \leq \ell_1 + 2$ ,  $\kappa = 1, 2, \dots$ . This follows from the fact that for  $i \geq m_j$  the first  $j - 1$  cycles commute with all the generators comprising  $(g_i)^{(p)}$  and, consequently, remain passive “spectators”. This feature will play a central role in the following section. The special cases of this expression that will be required in section 6, where we consider all the reduced traces appearing in  $H_2(q)$  up to  $H_7(q)$ , are dealt with in the present section. The general procedure is presented in the Appendix.

The identity

$$tr\left((g_1)^{(p)}\right) = q^{p-1} \sum_{i=0}^{p-1} \binom{p-1}{i} \left(\frac{q-1}{q}\right)^i tr\left((g_1)_{i+1}\right) \quad (5)$$

is obtained by straightforward application of the defining relations of the Hecke algebra, eq. 1, and of the properties of the trace, in particular  $tr(AB) = tr(BA)$ .

The recurrence relation

$$\begin{aligned} \text{tr}\left((g_1)_p(g_1)^{(\ell)}\right) &= f_{2p+1}\text{tr}\left((g_1)_p(g_{p+1})^{(\ell-p)}\right) \\ &+ (q-1)\sum_{\ell=1}^p q^\ell f_{2(p-\ell)+1}\text{tr}\left((g_1)_{\ell-1}(g_{\ell+1})_{p-\ell}(g_{p+1})^{(\ell-p)}\right) \end{aligned} \quad (6)$$

where  $f_p = \frac{q^p - (-1)^p}{q+1}$  and  $\ell > p$ , is demonstrated in the Appendix.

The simplest special case is

$$\begin{aligned} \text{tr}\left(g_1(g_1)^{(\ell)}\right) &= (q-1)\text{tr}\left((g_1)^{(\ell)}\right) + q\text{tr}\left(g_1(g_2)^{(\ell-1)}\right) \\ &= \frac{q^3+1}{q+1}\text{tr}\left(g_1(g_2)^{(\ell-1)}\right) + q(q-1)\text{tr}\left((g_2)^{(\ell-1)}\right) \end{aligned}$$

and the next is

$$\begin{aligned} \text{tr}\left((g_1)_2(g_1)^{(\ell)}\right) &= \frac{q^5+1}{q+1}\text{tr}\left((g_1)_2(g_3)^{(\ell-2)}\right) \\ &+ q(q-1)\frac{q^3+1}{q+1}\text{tr}\left(g_1(g_2)^{(\ell-2)}\right) + q^2(q-1)\text{tr}\left(g_1(g_3)^{(\ell-2)}\right) \end{aligned} \quad (7)$$

The recurrence relation

$$\text{tr}\left((g_1)_{\ell-1}(g_\ell)^{(k-\ell+1)}\right) = (q-1)\text{tr}\left((g_1)_\ell(g_{\ell+1})^{(k-\ell)}\right) + q\text{tr}\left((g_1)_{\ell-1}(g_\ell)^{(k-\ell)}\right)$$

is straightforwardly derived. It can be used to obtain the explicit expression

$$\text{tr}\left((g_1)_{\ell-1}(g_\ell)^{(m+1)}\right) = q^m \sum_{i=0}^m \binom{m}{i} \left(\frac{q-1}{q}\right)^i \text{tr}\left((g_1)_{\ell+i}\right). \quad (8)$$

Furthermore,

$$\text{tr}\left((g_1)_\ell(g_\ell)^{(m+1)}\right) = q^m \sum_{i=0}^m \left\{ (q-1)\binom{m}{i} + \frac{q}{q-1}\binom{m-1}{i-1} \right\} \left(\frac{q-1}{q}\right)^i \text{tr}\left((g_1)_{\ell+i}\right) \quad (9)$$

Finally, when  $k > 2$

$$\begin{aligned} \text{tr}\left((g_1)_3(g_2)^{(k)}\right) &= \frac{q^5+1}{q+1}\text{tr}\left((g_1)_3(g_4)^{(k-2)}\right) + q(q-1)\frac{q^3+1}{q+1}\text{tr}\left((g_1)_2(g_3)^{(k-2)}\right) \\ &+ q^2(q-1)\text{tr}\left(g_1 g_3(g_4)^{(k-2)}\right) \end{aligned} \quad (10)$$

Note that the expressions reduced in eqs. 7 and 10 involve the same number of overlapping generators, and, consequently, have the same structure. This property is fully elucidated in the Appendix.

## 6 Reduced traces in terms of traces of products of Murphy operators

It is a simple matter to construct the (diagonal) representation matrices of the Murphy operators, from which their traces, as well as those of products of arbitrary subsets of Murphy operators, can readily be obtained. In fact, we show in section 7 that the traces of products of Murphy operators can be directly evaluated by a very simple recursive procedure, so that the actual representation matrices need not be constructed.

In the present section we discuss the evaluation of reduced traces in terms of traces of products of Murphy operators. For the sake of clarity of the presentation we proceed by considering cases of ascending complexity, thereby introducing the procedures required in their simplest possible form. Our guideline in the present section is to develop explicit expressions for all the reduced traces appearing in Hecke algebras up to  $H_7(q)$ , inclusive. Once this is achieved the generalization to arbitrary Hecke algebras becomes obvious, in principle. Note, however, that the expression obtained for any particular reduced trace in terms of appropriate traces of Murphy operator products, within the lowest Hecke algebra within which that reduced trace appears, remains valid for the traces of that particular type for all  $H_n(q)$ .

Using eqs. 2 and 5 and the binomial identity  $\sum_{i=1}^{p-1-j} \binom{p-1-i}{j} = \binom{p-1}{j+1}$  we obtain

$$tr(L_k) = \sum_{i=0}^{k-2} \binom{k-1}{i+1} \left( \frac{q-1}{q} \right)^i tr((g_1)_{i+1})$$

Inverting, we obtain explicit expressions for all the reduced traces consisting of single cycles (singly connected sequences) in terms of traces of Murphy operators

$$tr((g_1)_{k-1}) = \left( \frac{q}{q-1} \right)^{k-2} \sum_{i=0}^{k-2} (-1)^i \binom{k-1}{i} tr(L_{k-i}) \quad (11)$$

To proceed, we formulate the following recursive procedure for the evaluation of reduced traces with  $\kappa + 1$  disjoint cycles, starting from the expressions for appropriate reduced traces with  $\kappa$  disjoint cycles. Let  $\Psi_{(\ell_1, \ell_2, \dots, \ell_\kappa)}$  be a product of  $\kappa$  disjoint cycles of lengths  $\ell_1, \ell_2, \dots, \ell_\kappa$ .

For convenience we assume that  $\ell_1 \leq \ell_2 \leq \dots \leq \ell_\kappa$  and that the representative product  $\Psi_{(\ell_1, \ell_2, \dots, \ell_\kappa)}$  is chosen with minimal spacings among consecutive cycles, *i.e.*,  $\Psi_{(\ell_1, \ell_2, \dots, \ell_\kappa)} = (g_1)_{\ell_1} (g_{\ell_1+2})_{\ell_2} (g_{\ell_1+\ell_2+3})_{\ell_3} \dots (g_{\ell_1+\ell_2+\dots+\ell_{\kappa-1}+\kappa})_{\ell_\kappa}$ . Let us assume that  $\Psi_{(\ell_1, \ell_2, \dots, \ell_\kappa)}$  has already been expressed as a linear combination of the traces of a certain set of Murphy operators

$$\text{tr}(\Psi_{(\ell_1, \ell_2, \dots, \ell_\kappa)}) = \sum_{(i_1, i_2, \dots, i_\kappa)} C_{i_1, i_2, \dots, i_\kappa}^{(\ell_1, \ell_2, \dots, \ell_\kappa)} \text{tr}(L_{i_1} L_{i_2} \dots L_{i_\kappa}) \quad (12)$$

where  $0 \leq i_1 \leq i_2 \leq \dots \leq i_\kappa$  and where  $L_0 \equiv 1$ . To obtain the reduced traces with  $\kappa + 1$  cycles of lengths  $\ell_1, \ell_2, \dots, \ell_\kappa, \ell_{\kappa+1}$ , where, without loss of generality, we assume that  $\ell_{\kappa+1} \geq \ell_\kappa$ , we note

LEMMA 4:  $\text{tr}(\Psi_{(\ell_1, \ell_2, \dots, \ell_\kappa)} L_m) = \sum_{(i_1, i_2, \dots, i_\kappa)} C_{i_1, i_2, \dots, i_\kappa}^{(\ell_1, \ell_2, \dots, \ell_\kappa)} \text{tr}(L_{i_1} L_{i_2} \dots L_{i_\kappa} L_m)$ , where  $m > \ell_1 + \ell_2 + \dots + \ell_\kappa + \kappa$ .

PROOF: Since  $L_m$  commutes with all the generators comprising  $\Psi_{(\ell_1, \ell_2, \dots, \ell_\kappa)}$ , all the manipulations leading to the identity 12 can be carried out through  $L_m$ .

The remaining task is the reduction of  $\text{tr}(\Psi_{(\ell_1, \ell_2, \dots, \ell_\kappa)} L_m)$  into a linear combination of reduced traces. This is effected using the recurrence relations presented in section 5 and in the Appendix. Assuming that all traces with up to  $\kappa$  disjoint cycles, as well as all traces with  $\kappa + 1$  disjoint cycles such that the first  $\kappa$  cycles precede  $\Psi_{(\ell_1, \ell_2, \dots, \ell_\kappa)}$  in a lexicographic ordering, have been determined, we begin with  $m = \ell_1 + \ell_2 + \dots + \ell_\kappa + \kappa + \ell_\kappa + 1$  to obtain the new trace with  $\ell_{\kappa+1} = \ell_\kappa$ , *i.e.*,  $\text{tr}(\Psi_{(\ell_1, \ell_2, \dots, \ell_\kappa, \ell_\kappa)})$ . We then increase  $m$  to obtain the reduced traces with higher  $\ell_{\kappa+1}$ .

Thus, from eq. 11 we obtain

$$\text{tr}((g_1)_{k-1} L_m) = \left( \frac{q}{q-1} \right)^{k-2} \sum_{i=0}^{k-2} (-1)^i \binom{k-1}{i} \text{tr}(L_{k-i} L_m). \quad (13)$$

where  $m > k + 1$ . Note that all the terms on the right hand side are traces of products of Murphy operators.

To proceed, the left hand side of eq. 13 has to be expressed as a linear combination of reduced traces. By a judicious choice of  $m$  precisely one of these will be of a new type. That trace would be the one that the resulting identity would allow evaluating.

The simplest case is  $tr(g_1 L_m)$ , for which eq. 13 is trivial, yielding  $tr(g_1 L_m) = tr(L_2 L_m)$ .

To evaluate that trace we need the three identities

$$tr(g_1 (g_3)^{(p-2)}) = q^{p-3} \sum_{i=0}^{p-3} \binom{p-3}{i} \left( \frac{q-1}{q} \right)^i tr(g_1 (g_3)_{i+1}) ,$$

which, like eq. 13, follows from the fact that  $g_1$  commutes with all the generators comprising  $(g_3)^{(p-2)}$ ,

$$tr(g_1 (g_2)^{(p-1)}) = q^{p-2} \sum_{i=0}^{p-2} \binom{p-2}{i} \left( \frac{q-1}{q} \right)^i tr((g_1)_{i+2})$$

which is a special case of eq. 8, and

$$tr(g_1 (g_1)^{(p)}) = q^{p-1} \sum_{i=0}^{p-1} \left\{ (q-1) \binom{p-1}{i} + \frac{q}{q-1} \binom{p-1}{i-1} \right\} \left( \frac{q-1}{q} \right)^i tr((g_1)_{i+1})$$

which is a special case of eq. 9.

Thus, for  $m \geq 4$

$$\begin{aligned} tr(g_1 L_m) &= \sum_{j=0}^{m-4} \left( \frac{q-1}{q} \right)^j \binom{m-3}{j+1} tr(g_1 (g_3)_{j+1}) \\ &+ \sum_{j=0}^{m-2} \left\{ \frac{2q}{q-1} \binom{m-3}{j-1} + (q-1) \binom{m-2}{j} \right\} \left( \frac{q-1}{q} \right)^j tr((g_1)_{j+1}) \end{aligned} \quad (14)$$

In fact, for  $m = 3$  eq. 14 reduces to

$$tr(g_1 L_3) = (q-1)tr(g_1) + \left( q + \frac{1}{q} \right) tr(g_1 g_2) ,$$

that provides no new information. Expressing the traces on the right-hand-side by means of  $tr(L_2)$  and  $tr(L_3)$  we obtain the identity

$$(q-1)tr(L_2 L_3) = (q^2 + 1)tr(L_3) - (q+1)^2 tr(L_2) .$$

Eq. 14 enables the systematic evaluation of all reduced traces of the form  $tr(g_1 (g_3)_j)$  ;  $j = 1, 2, \dots$ , *i.e.*, the traces of all elements consisting of a cycle of length two and a disjoint cycle of arbitrary length.

As an example consider

$$tr(g_1 L_4) = tr(g_1 g_3) + \frac{q-1}{q} (q + \frac{1}{q}) tr((g_1)_3) + 2(q-1 + \frac{1}{q}) tr((g_1)_2) + (q-1)tr(g_1) \quad (15)$$

which enables the evaluation of the non-simply-connected element  $tr(g_1 g_3)$  consisting of two disjoint cycles of unit length, *i.e.*,

$$tr(g_1 g_3) = tr(L_2 L_4) - \frac{q^2 + 1}{q - 1} tr(L_4) + \frac{(q + 1)^2}{q - 1} tr(L_3) - \frac{2q}{q - 1} tr(L_2) . \quad (16)$$

As a further example consider  $tr(g_1 L_5)$ , that can be expressed in terms of  $tr(g_1 g_3)$ ,  $tr(g_1 g_3 g_4)$ , and traces of simply connected terms. Thus,

$$\begin{aligned} tr(g_1 L_5) &= 2 tr(g_1 g_3) + \left( \frac{q - 1}{q} \right) tr(g_1 g_3 g_4) \\ &+ \left( \frac{q - 1}{q} \right)^2 \left( q + \frac{1}{q} \right) tr(g_1 g_2 g_3 g_4) + \left( \frac{q - 1}{q} \right) \left( 3q - 2 + \frac{3}{q} \right) tr(g_1 g_2 g_3) \\ &+ \left( 3q - 4 + \frac{3}{q} \right) tr(g_1 g_2) + (q - 1) tr(g_1) \end{aligned}$$

Since  $tr(g_1 g_3)$ , as well as all the simply-connected traces, have already been evaluated,  $tr(g_1 L_5)$  provides the next non-simply connected term,  $tr(g_1 g_3 g_4)$ .

Next, we consider  $tr(g_1 g_2 L_m)$ , for which eq. 13 yields

$$tr(g_1 g_2 L_m) = \frac{q}{q - 1} \left( tr(L_3 L_m) - 2 tr(L_2 L_m) \right) .$$

Since the right hand side consists of traces of products of Murphy operators, the remaining task is the reduction of the left hand side. We obtain

$$\begin{aligned} tr(g_1 g_2 L_m) &= \frac{1}{q^{m-2}} tr((g_1)_2 (g_1)^{(m-1)}) + \frac{1}{q^{m-3}} tr((g_1)_2 (g_2)^{(m-2)}) \\ &+ \frac{1}{q^{m-4}} tr((g_1)_2 (g_3)^{(m-3)}) + \sum_{i=4}^m \frac{1}{q^{m-1-i}} tr((g_1)_2 (g_4)^{(m-i)}) \end{aligned}$$

where the first, second and third terms on the right hand side are special cases of eqs. 6, 9 and 8, respectively. All these terms reduce into single cycle traces, with the exception of the contribution  $tr((g_1)_2 (g_3)^{(m-3)})$  to the first term. Note, however, that this trace was already evaluated in terms of  $tr(L_2 L_m)$ . Each summand in the fourth term can be reduced into two-cycle traces. Thus, using eq. 5 and the fact that  $(g_1)_2$  commutes with all the generators comprising  $(g_4)^{(j)}$ ,

$$tr((g_1)_2 (g_4)^{(j)}) = q^{j-1} \sum_{i=0}^{j-1} \binom{j-1}{i} \left( \frac{q-1}{q} \right)^i tr((g_1)_2 (g_4)_{i+1}) .$$



Hence,  $tr(g_1g_2L_4)$  and  $tr(g_1g_2L_5)$  consist of reduced traces that we already evaluated, and yield no new information.  $tr(g_1g_2L_6)$  gives rise to the new doubly connected term  $tr(g_1g_2g_4g_5)$ . In general,  $tr(g_1g_2L_m)$  yields, in addition to known terms, the new term  $tr((g_1)_2(g_4)_{m-4})$ .

For  $tr(g_1g_2g_3L_m)$  we obtain from eq. 13

$$tr(g_1g_2g_3L_m) = \left(\frac{q}{q-1}\right)^2 \left(tr(L_4L_m) - 3tr(L_3L_m) + 3tr(L_2L_m)\right)$$

On the other hand,

$$\begin{aligned} tr(g_1g_2g_3L_m) &= \frac{1}{q^{m-1}}tr((g_1)_3(g_1)^{(m)}) + \frac{1}{q^{m-2}}tr((g_1)_3(g_2)^{(m-1)}) \\ &+ \frac{1}{q^{m-3}}tr((g_1)_3(g_3)^{(m-2)}) + \frac{1}{q^{m-4}}tr((g_1)_3(g_4)^{(m-3)}) \\ &+ \sum_{i=5}^m \frac{1}{q^{m-i}}tr((g_1)_3(g_i)^{(m-i)}) \end{aligned}$$

New reduced terms only appear for  $m \geq 8$ , consisting of a sequence of length three (*i.e.*, a cycle of length *four*), and a disjoint sequence of length  $m-5$ . For  $m=6,7$  we obtain reduced traces with two disjoint sequences of lengths  $3+1$  and  $3+2$ , respectively, that have already been evaluated above. The first four terms are evaluated using eqs. 6, 10, 9, and 8, respectively.

From eq. 16

$$tr(g_1g_3L_6) = tr(L_2L_4L_6) - \frac{q^2+1}{q-1}tr(L_4L_6) + \frac{(q+1)^2}{q-1}tr(L_3L_6) - \frac{2q}{q-1}tr(L_2L_6). \quad (17)$$

Now,

$$\begin{aligned} tr(g_1g_3L_6) &= tr(g_1g_3g_5) + \frac{1}{q}tr(g_1g_3(g_4)^{(2)}) + \frac{1}{q^2}tr(g_1g_3(g_3)^{(3)}) \\ &+ \frac{1}{q^3}tr(g_1g_3(g_2)^{(4)}) + \frac{1}{q^4}tr(g_1g_3(g_1)^{(5)}) \end{aligned} \quad (18)$$

Only the last two terms on the right hand side do not immediately follow from identities previously derived. Presenting them in a slightly generalized form we obtain

$$tr(g_1g_3(g_2)^{(\ell)}) = (q-1)(q^2+1)tr((g_1)_3(g_4)^{(\ell-2)}) + (q-1)^2q tr((g_1)_2(g_3)^{(2)}) + q^2tr(g_1g_3(g_4)^{(2)})$$

and

$$\begin{aligned}
tr(g_1 g_3 (g_1)^{(\ell)}) &= (q-1)(q^2+1)(q^2-q+1)tr((g_1)_3 (g_4)^{(\ell-3)}) \\
&+ (q-1)^2 q(2q^2-q+2)tr((g_2)_2 (g_4)^{(\ell-3)}) + q^2(q^2-q+1)tr(g_1 g_3 (g_4)^{(\ell-3)}) \\
&+ q^2(q-1)^3 tr(g_3 (g_4)^{(\ell-3)}) + (q-1)q^3 tr(g_2 (g_4)^{(\ell-3)})
\end{aligned}$$

Hence, eq. 18 provides the triply connected trace  $tr(g_1 g_3 g_5)$ .

Similarly, replacing  $L_6$  by  $L_7$  in eq. 17 we can derive an expression for the reduced trace  $tr(g_1 g_3 g_5 g_6)$ .

The explicit expressions for all the reduced traces appearing up to  $H_7(q)$ , in terms of traces of products of Murphy operators, are presented in Table 1.

## 7 Evaluation of the traces of products of Murphy operators within irreps of $H_n(q)$

In the previous section we have expressed the reduced traces in terms of traces of products of Murphy operators. We now present a recursive procedure that can be implemented to obtain the latter traces.

For an irrep  $\Gamma_n$  of  $H_n(q)$  consider the set of irreps  $\Gamma_{n-1} \subset \Gamma_n$  of  $H_{n-1}(q)$ , obtained by eliminating one box from  $\Gamma_n$  in all possible ways. Clearly,

$$tr(L_i)_{\Gamma_n} = \sum_{\Gamma_{n-1} \subset \Gamma_n} tr(L_i)_{\Gamma_{n-1}} \quad i = 2, 3, \dots, n-1.$$

and

$$|\Gamma_n| = \sum_{\Gamma_{n-1} \subset \Gamma_n} |\Gamma_{n-1}|$$

where  $|\Gamma_n|$  is the dimensionality of the irrep  $\Gamma_n$ .

The trace of  $L_n$  can now be evaluated very conveniently using

$$tr(L_n)_{\Gamma_n} = \sum_{\Gamma_{n-1} \subset \Gamma_n} |\Gamma_{n-1}| \{\Gamma_n \setminus \Gamma_{n-1}\}_q.$$

We proceed to obtain the traces of products of distinct, non-consecutive Murphy operators. For  $\prod_{i=1}^{\ell} L_{\alpha_i}$ ,  $\alpha_1 \geq 2$ ,  $\alpha_{i+1} \geq \alpha_i + 2$ ,  $i = 1, 2, \dots, \ell - 1$ , and  $\alpha_{\ell} \leq n$ , we consider all the sequences of irreps leading to the desired irrep  $\Gamma_n$  of  $H_n(q)$  from all possible irreps of  $H_{\alpha_1-1}(q)$ . Let  $\Gamma_{\alpha_1-1} \subset \Gamma_{\alpha_1} \subset \Gamma_{\alpha_1+1} \subset \dots \subset \Gamma_n$  be such a sequence and let  $\{\dots \subset \Gamma_n\}$  denote the complete set of relevant sequences. Obviously,

$$tr \left( \prod_{i=1}^{\ell} L_{\alpha_i} \right)_{\Gamma_n} = \sum_{\{\dots \subset \Gamma_n\}} |\Gamma_{\alpha_1-1}| \prod_{i=1}^{\ell} \{\Gamma_{\alpha_i} \setminus \Gamma_{\alpha_i-1}\}_q.$$

This expression can be rewritten in the following form:

if  $\alpha_{\ell} < n$ :

$$tr \left( \prod_{i=1}^{\ell} L_{\alpha_i} \right)_{\Gamma_n} = \sum_{\Gamma_{n-1} \subset \Gamma_n} tr \left( \prod_{i=1}^{\ell} L_{\alpha_i} \right)_{\Gamma_{n-1}}$$

if  $\alpha_{\ell} = n$ :

$$tr \left( \prod_{i=1}^{\ell} L_{\alpha_i} \right)_{\Gamma_n} = \sum_{\Gamma_{n-1} \subset \Gamma_n} \{\Gamma_n \setminus \Gamma_{n-1}\}_q tr \left( \prod_{i=1}^{\ell-1} L_{\alpha_i} \right)_{\Gamma_{n-1}}$$

Hence, given the traces of all Murphy operators up to  $L_{n-2}$  and of all products of sets of non-consecutive Murphy operators, corresponding to the irreps of  $H_{n-1}(q)$ , those corresponding to the irreps of  $H_n(q)$  are readily obtained.

The procedure presently described is ideally suited to implementation using symbolic programming. Such an implementation was used to evaluate the traces of all relevant products of Murphy operators in  $H_6(q)$  and  $H_7(q)$ , in terms of which the corresponding reduced traces were obtained. The results for  $H_6(q)$  agree with those in ref. [2], and the results for  $H_7(q)$  are presented in table 2. For  $q = 1$  they reduce to the characters of the symmetric group  $S_7$ . As a further check we note that the traces of reduced elements consisting of  $\ell$  generators, that correspond to conjugate Young diagrams, are related by the transformation  $q^{\ell-i} \mapsto (-1)^{\ell} q^i$ ,  $i = 0, 1, \dots, \ell$ .

## 8 Conclusions

In the present article we have demonstrated that the trace of an arbitrary element of the Hecke algebra  $H_n(q)$  can be reduced in a systematic way into a linear combination of reduced

traces, consisting of disjoint sequences of consecutive generators. The reduced traces depend only on their cycle structures, *i.e.*, on the set of lengths of their disjoint sequences, and can be expressed in terms of linear combinations of traces of products of Murphy operators. The latter traces can be evaluated using a straightforward recursive procedure.

One favourable feature of this approach is that the expression for a given type of reduced trace in terms of traces of products of Murphy operators does not explicitly depend on  $n$ . Consequently, the present procedure is particularly attractive when specific types of traces, *e.g.*, traces with some restriction on the set of disjoint cycles, are required to high order in  $n$ . Thus, the present scheme may be of interest towards the investigation of the high  $n$  limit of restricted classes of reduced traces, a problem of some current interest [11].

## Acknowledgements

Helpful discussions with Alain Lascoux, Bernard Leclerc, Jean-Yves Thibon and Gérard Duchamp are gratefully acknowledged.

## APPENDIX : The trace reduction procedure.

In this appendix we show how to obtain the reduced form of the trace

$$\begin{aligned} & tr \left( (g_1 \dots g_{m_1-1})(g_{m_1+1} \dots g_{m_2-1}) \dots (g_{m_j+1} \dots g_p)(g_k \dots g_{p+r} \dots g_k) \right) \\ & \equiv tr \left( (g_1)_{m_1-1} (g_{m_1+1})_{m_2-m_1-1} \dots (g_{m_j+1})_{p-j} (g_k)^{p+r-k+1} \right) \end{aligned} \quad (19)$$

where  $(g_1)_{m_1-1}(g_{m_1+1})_{m_2-m_1-1} \dots (g_{m_j+1})_{p-j}$  is a reduced element consisting of  $j+1$  connected sequences, or, in other words, a sequence  $g_1 g_2 \dots g_p$  with  $j$  cuts at  $m_i$ ,  $i = 1, \dots, j$ , and  $(g_k)^{(p+r-k+1)}$ ,  $1 \leq k \leq p+1$ , is a term contributed by the Murphy operator  $L_{p+r+1}$ . It is shown in section 6 that the reduction of eq. 19 is sufficient in order to express all reduced traces in terms of the traces of products of Murphy operators.

A central role is played by the coefficients  $f_p$  defined through

$$g_i^p = f_p g_i + q f_{p-1}$$

They satisfy the recurrence relation

$$f_p = (q-1)f_{p-1} + q f_{p-2} \quad (20)$$

with  $f_0 \equiv 0$  and  $f_1 = 1$ , that yields

$$f_p = \frac{q^p - (-1)^p}{q+1} = q^{p-1} - q^{p-2} + \dots + (-1)^{p-1} \quad (21)$$

In terms of these we will obtain *f-expansions* leading to completely reduced forms.

Systematic use will be made of the lemmas

$$g_i(g_j g_{j+1} \dots g_k) = (g_j g_{j+1} \dots g_k) g_{i-1} \quad (j < i \leq k) \quad (22)$$

and

$$[g_i, (g_j \dots g_{j+r} \dots g_j)] = 0 \quad (j < i < j+r) \quad (23)$$

We start with the simplest form of eq. 19, allowing no cuts ( $j = 0$ ), proceeding step-wise up in the order of complexity.

Define

$$V_k^{(p,r)} \equiv \text{tr}((g_1 \dots g_{k-1})(g_{k+1} \dots g_p)(g_{p+1} \dots g_{p+r} \dots g_{p+1})) \quad ; \quad 0 \leq k \leq p-1$$

where  $(g_i)_0 \equiv 1$  so that

$$V_0^{(p,r)} \equiv \text{tr}((g_1 \dots g_p)(g_{p+1} \dots g_{p+r} \dots g_{p+1}))$$

Using 22 and 23 one easily obtains the reduced form

$$V_k^{(p,r)} = \sum_{\ell=0}^{r-1} \binom{r-1}{\ell} q^\ell (q-1)^{r-\ell-1} \text{tr}((g_1 \dots g_{k-1})(g_{k+1} \dots g_{p+r-\ell}))$$

For fixed  $p$  and  $r$ , when there is no risk of confusion, we will write  $V_k$  for  $V_k^{(p,r)}$ . Consider now the case with  $p-l+1$  overlapping generators, but no cut, namely

$$\text{tr}((g_1 \dots g_p)(g_l \dots g_{p+r} \dots g_l)) \quad (\ell \leq p)$$

and note that for  $l = p+1$  one has just  $V_0^{(p,r)}$ , and for  $l > p+1$  the reduced form is obtained trivially.

For  $p \geq 2l-1$ , repeatedly using the lemmas 22 and 23 along with cyclic permutations within the trace, one obtains

$$\begin{aligned} & \text{tr}((g_1 \dots g_p)(g_l \dots g_{p+r} \dots g_l)) \\ &= \text{tr}((g_{p-2k} \dots g_{l-k})(g_1 \dots g_p)(g_{l-k} \dots g_p)(g_{p+1} \dots g_{p+r} \dots g_{p+1})) \quad (k \leq l-1) \\ &= \text{tr}((g_{p-l+1} \dots g_1)(g_1 \dots g_{p-l+1})(g_1 \dots g_p)(g_{p+1} \dots g_{p+r} \dots g_{p+1})) \end{aligned} \quad (24)$$

The final form is well defined for all  $p \geq l$ , but the intermediate steps make the inequality  $p \geq 2l-1$  necessary.

Using eq. 3

$$(g_{p-l+1} \dots g_1)(g_1 \dots g_{p-l+1}) = (q-1) \sum_{k=1}^{p-l+1} q^{k-1} (g_k \dots g_{p-l+1} \dots g_k) + q^{p-l+1}$$

and, as follows from eq. 22,

$$\begin{aligned} & \text{tr}((g_k \dots g_{p-l+1} \dots g_k)(g_1 \dots g_p)(g_{p+1} \dots g_{p+r} \dots g_{p+1})) \\ &= \text{tr}((g_1 \dots g_{p-l-k+2} \dots g_1)(g_1 \dots g_p)(g_{p+1} \dots g_{p+r} \dots g_{p+1})) \end{aligned}$$

we obtain from 24

$$tr\left((g_1 \dots g_p)(g_l \dots g_{p+r} \dots g_l)\right) = (q-1) \sum_{r=1}^{p-l+1} q^{r-1} A_{p-l-r+2}^{(p,r)} + q^{p-l+1} V_0^{(p,r)} \quad (25)$$

where

$$A_s^{(p,r)} \equiv tr\left((g_1 \dots g_s \dots g_1)(g_1 \dots g_p)(g_{p+1} \dots g_{p+r} \dots g_{p+1})\right) \quad (26)$$

Skipping details and suppressing the superscript  $(p, r)$  we state the recurrence relation

$$A_s = (q-1)sq^{s-1}V_0 + q^s V_s + (q-1)^2 \sum_{r=1}^{s-1} r q^{r-1} A_{s-r} \quad (27)$$

In the derivation an auxiliary construction

$$X_{(k,l)} \equiv tr\left((g_1 \dots g_k)(g_{k+1} \dots g_l \dots g_{k+1})(g_k \dots g_p)(g_{p+1} \dots g_{p+r} \dots g_{p+1})\right),$$

satisfying

$$X_{(k,l)} = (q-1)A_{l-k} + qX_{(k,l-1)},$$

was used. Eq. 27 is satisfied by

$$A_s = f_{2s}V_0 + (q-1) \sum_{m=1}^{s-1} q^m f_{2(s-m)}V_m + q^s V_s \quad (28)$$

This follows, by induction, noting that

$$\begin{aligned} A_1 &= tr\left(g_1(g_1 \dots g_p)(g_{p+1} \dots g_{p+r} \dots g_{p+1})\right) \\ &= (q-1)V_0 + qV_1 \end{aligned}$$

and that, using 21 and summing the series in  $k$ , one obtains the identity

$$(q-1)^2 \sum_{k=1}^{p-1} k q^{k-1} f_{2(p-k)} + (q-1)p q^{p-1} = f_{2p}, \quad (29)$$

with the aid of which the induction is straightforward. Finally, injecting 28 in 25 and using the identity (obtained like 29),

$$(q-1) \sum_{\ell=1}^p q^{\ell-1} f_{2(p+1-\ell)} + q^p = f_{2p+1}$$

one obtains,

$$tr\left((g_1 \dots g_p)(g_l \dots g_{p+r} \dots g_l)\right) = f_{2(p-l)+3}V_0 + (q-1) \sum_{k=1}^{p-l+1} q^k f_{2(p-l-k)+3}V_k \quad (30)$$

In the partial reduction 24 we had to impose  $p \geq 2l - 1$ . For the complementary situation ( $p < 2l - 1$ ) *the same final result*, eq. 30, can be obtained following a different route. This will now be briefly indicated. One can show, using the lemmas along with cyclic permutations, that

$$\begin{aligned} & tr \left( (g_1 \dots g_p)(g_l \dots g_{p+r} \dots g_l) \right) \\ &= tr \left( (g_1 \dots g_{2(p-l+1)})(g_1 \dots g_p)(g_{p+1} \dots g_{p+r} \dots g_{p+1}) \right) \end{aligned} \quad (31)$$

Define

$$B_k = tr \left( (g_1 \dots g_k)(g_1 \dots g_p)(g_{p+1} \dots g_{p+r} \dots g_{p+1}) \right)$$

The following recurrence relations have been obtained

$$B_{2(k+1)} = q^{k+1}V_0 + (q-1) \sum_{r=0}^k q^{k-r} B_{2r+1} \quad (32)$$

$$B_{2k+1} = q^{k+1}V_{k+1} + (q-1) \sum_{r=0}^k q^{k-r} B_{2r} \quad (33)$$

In the derivation we used the auxiliary construction

$$Y_{(k,s)} \equiv tr \left( (g_1 \dots g_{k-s})(g_{s+1} \dots g_p)(g_{p+1} \dots g_{p+r} \dots g_{p+1}) \right) \quad (k \geq 2s + 1, p > k)$$

that satisfies

$$Y_{(k,s)} = (q-1)B_{(k-2s-1)} + qY_{(k,s+1)}$$

and

$$B_k = (q-1)B_{k-1} + qY_{(k,1)} .$$

The solutions of 32 and 33 are

$$B_{2k} = f_{2k+1}V_0 + (q-1) \left( \sum_{m=1}^k q^m f_{2(k-m)+1}V_m \right)$$



$$B_{2k+1} = f_{2k+2}V_0 + (q-1)\left(\sum_{m=1}^k q^m f_{2(k-m+1)}V_m\right) + q^{k+1}V_{k+1}$$

Since the right hand side of 31 is just  $B_{2(p-l+1)}$  we get eq. 30 again, as promised. Hence, the crucial data (both for  $p \geq 2l-1$  and for  $p < 2l-1$ ) determining the structure of the  $f$ -expansion, eq. 30, is the length of the overlap  $p-l+1$ . This is reminiscent of, but distinct from, the result that the reduced traces depend only on the lengths of the disjoint cycles.

We now generalize to 19 with a single cut, namely

$$tr\left((g_1 \dots g_{k-1})(g_{k+1} \dots g_p)(g_l \dots g_{p+r} \dots g_l)\right)$$

**Case 1.**  $l > k$

The generalization is trivial. The factor  $(g_1 \dots g_{k-1})$  commutes with all the generators in the other two factors, that can be reduced essentially as in the previous case (without cut), with  $(g_1 \dots g_{k-1})$  as a passive spectator.

**Case 2.**  $l = k$

One has

$$\begin{aligned} & tr \left( (g_1 \dots g_p)(g_l \dots g_{p+r} \dots g_l) \right) \\ &= tr \left( (g_1 \dots g_l)(g_l \dots g_{p+r} \dots g_l)(g_{l+1} \dots g_p) \right) \\ &= (q-1)tr \left( (g_1 \dots g_{l-1})(g_{l+1} \dots g_p)(g_l \dots g_{p+r} \dots g_l) \right) + q tr \left( (g_1 \dots g_p)(g_{l+1} \dots g_{p+r} \dots g_{l+1}) \right) \end{aligned}$$

Hence,

$$\begin{aligned} & tr \left( (g_1 \dots g_{l-1})(g_{l+1} \dots g_p)(g_l \dots g_{p+r} \dots g_l) \right) \\ &= \frac{1}{q-1} \left\{ tr \left( (g_1 \dots g_p)(g_l \dots g_{p+r} \dots g_l) \right) - q tr \left( (g_1 \dots g_p)(g_{l+1} \dots g_{p+r} \dots g_{l+1}) \right) \right\} \quad (34) \\ &= f_{2(p-l-1)}V_0 + (q-1) \sum_{k=1}^{p-l} q^k f_{2(p-l-k+1)}V_k + q^{p-l+1}V_{p-l+1} \end{aligned}$$

where we used eqs. 30 and 20.

An evident generalization of eq. 34 is

$$\begin{aligned}
& tr \left( (g_1 \dots g_{m-1})(g_{m+1} \dots g_{l-1})(g_{l+1} \dots g_p)(g_l \dots g_{p+r} \dots g_l) \right) \\
&= \frac{1}{(q-1)} \left\{ tr \left( (g_1 \dots g_{m-1})(g_{m+1} \dots g_p)(g_l \dots g_{p+r} \dots g_l) \right) \right. \\
&\quad \left. - q tr \left( (g_1 \dots g_{m-1})(g_{m+1} \dots g_p)(g_{l+1} \dots g_{p+r} \dots g_{l+1}) \right) \right\}
\end{aligned} \tag{35}$$

An analogous result holds quite generally for  $k = m_j$  in 19.

The results 34, 35 and their evident successive generalizations will be used repeatedly for the cases to follow.

**Case 3.**  $l < k$

Now using lemma 23

$$tr \left( (g_1 \dots g_{k-1})(g_{k+1} \dots g_p)(g_l \dots g_{p+r} \dots g_l) \right) = tr \left( (g_1 \dots g_{k-1})(g_l \dots g_{p+r} \dots g_l)(g_{k+1} \dots g_p) \right) \tag{36}$$

One can reduce the first two factors on the right hand side of eq. 36, leading to 30, with  $(g_{k+1} \dots g_p)$  as spectator. This gives

$$\begin{aligned}
& tr \left( (g_1 \dots g_{k-1})(g_{k+1} \dots g_p)(g_l \dots g_{p+r} \dots g_l) \right) \\
&= f_{2(k-l)+1} U_0 + (q-1) \sum_{s=1}^{k-l} q^s f_{2(k-l-s)+1} U_s
\end{aligned}$$

where

$$\begin{aligned}
U_m &= tr \left( (g_1 \dots g_{m-1})(g_{m+1} \dots g_{k-1})(g_k \dots g_{p+r} \dots g_k)(g_{k+1} \dots g_p) \right) \\
&= tr \left( (g_1 \dots g_{m-1})(g_{m+1} \dots g_{k-1})(g_{k+1} \dots g_p)(g_k \dots g_{p+r} \dots g_k) \right)
\end{aligned} \tag{37}$$

In particular

$$U_0 = tr \left( (g_1 \dots g_{k-1})(g_{k+1} \dots g_p)(g_k \dots g_{p+r} \dots g_k) \right)$$

has already been studied in case 2.

$$\begin{aligned}
U_1 &= tr \left( (g_2 \dots g_{k-1})(g_k \dots g_{p+r} \dots g_k)(g_{k+1} \dots g_p) \right) \\
&= tr \left( (g_1 \dots g_{k-2})(g_k \dots g_{p-1})(g_{k-1} \dots g_{p+r-1} \dots g_{k-1}) \right)
\end{aligned}$$

is also directly reduced to the preceding case.

For  $U_m(m > 1)$  one first uses eq. 35 to obtain traces with a single cut. After this step, since  $s \leq k - l < k$ , all the cuts in the traces  $U_s$  in eq. 37 lie below the overlap, as for case 1. Hence, one can at that stage obtain complete reduction.

*This process indicates clearly the generalization of the iterative process necessary for multiple cuts, as in 19.* One starts by shifting the factor  $(g_{m_j+1} \dots g_p)$  on the right and uses for the remaining factors the reduction procedure for  $j - 1$  cuts with known results. Finally, one proceeds as for the passage from  $j = 0$  to  $j = 1$ .

For the simple case 2 the  $f$ 's were recombined to obtain the simple form (35). A possible generalization of such combinatorics for arbitrary cuts is beyond the scope of this paper.

To sum up, we have the complete machinery permitting a multiple stage  $f$ -expansion of 19 to its reduced form. A whole set of interesting algebraic structures were discovered in the process.

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